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FRACTALS: HUNTING FOR FRACTIONAL DIMENSION AND ATTRACTIVE DESIGN

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Abstract. The article is devoted to fractal theory, outlined from its inception to today. It is shown that there are three aspects of fractals — mathematics, geometry, and beauty — and the details for each of these aspects is given.

Keywords: *fractal, self-similarity, Koch snowflake, coastline.*

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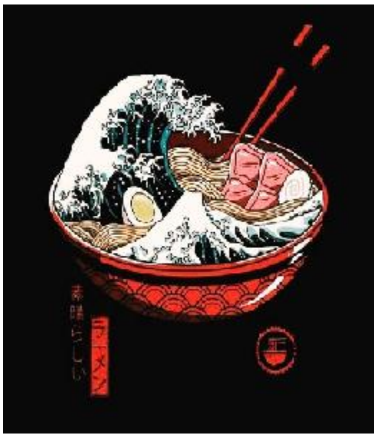
1. Introduction

For the whole of human history mathematics was used to describe and to explain phenomena and objects in nature. Nevertheless, there are many objects in nature, such as mountains, coastlines, tree branches, bark, clouds, snowfalls, coastlines, lightning, etc. that are too irregular to be described and analyzed by traditional geometry. That is the reason why fractal theory emerged and developed.

A good example of a fractal in nature is the Romanesco broccoli, showing self-similar form.

One of the most vivid examples of fractals in fine art, long before fractal theory emerged, are the paintings of Katsushika Hokusai (1760-1849). Hokusai's distinguished merit is that he noticed fractals in nature and portrayed them. His woodblock "The great wave off Kanagawa" is the most popular oriental piece of art (compared to something like the Mona Lisa in Europe) and it is popularized more and more by being printed on t-shirts, bags, magnets, mugs, etc. Hokusai's ideas were used in oriental restaurant ads and even in motherboard ads.


The great wave off Kanagawa Mount Fuji with cranes




Oriental restaurant ads

Motherboard ads.

2. Fractals in math: some simple examples

The simplest example of a fractal in mathematics may be an infinitely decreasing geometric progression, i.e. the sequence of numbers, such as a, $a\cdot q$, $a\cdot q^2$, $a\cdot q^3$,... and so on up to infinity, assuming that $q < 1$. It is easy to note the fractal structure of such a sequence. Imagine that we discard the first term, so we will have the sequence $a\cdot q$, $a\cdot q^2$, $a\cdot q^3$,... for which each term is 1/q times smaller, than the corresponding term of a, $a\cdot q$, $a\cdot q^2$, $a\cdot q^3$,... so we get a "small copy" of the initial sequence. The sum of such terms, i.e. $S = a+a\cdot q+a\cdot q^2+a\cdot q^3+...$ is the sum of an infinitely decreasing geometric progression. If $q < 1$, then despite S containing an infinite number of terms, the sum of such terms is finite.

This can be demonstrated in a visual way in the particular case where q = 1/2. Imagine you have two apples. Put one apple on the table. Cut the second one into two even parts and put the half next to the whole. Cut the other half into two quarters and put the quarter next to the half, and do this in the same way over and over again to get the sequence 1, 1/2, 1/4, 1/8,..... Since we had two apples at the very beginning, then it is obvious, that 1+1/2+1/4+1/8+...=2.

In order to calculate S in general case, one has to notice, that

$$S - a = a\cdot q+a\cdot q^2+a\cdot q^3+... = q(a+a\cdot q+a\cdot q^2+a\cdot q^3+...) = qS, \text{ so } S = \frac{a}{1-q}.$$

The next two cases are based on the same idea: to notice the fractal structure of the expression, namely, to notice that there is a part which is the same as the whole.

Example 1. Find S, where $S = \sqrt{2+\sqrt{2+\sqrt{2+...(up\ to\ \infty)}}}$.

This expression seems at first sight to be a "monster". But infinity gives us an opportunity to calculate it. Note that the internal part of the expression (after 2+ under the first root) is the same as the whole, so, $x = \sqrt{2+x}$, hence $x^2 = x+2$, $x \geq 0$, so that x = 2.

Example 2. Find S, where $S = 1 + \cfrac{1}{1+\cfrac{1}{1+...(up\ to\ \infty)}}$.

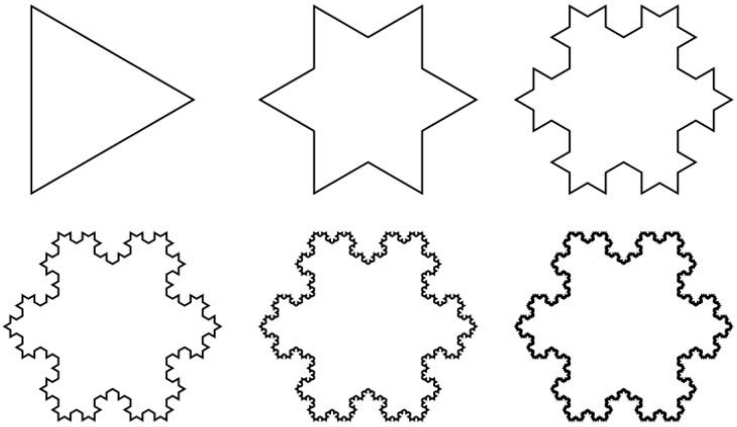
Similar to the previous case, one notes that the denominator is the same as the whole expression, so that $S = 1+\frac{1}{S}$, and therefore $S^2 - S - 1 = 0$.

There are two roots of such an equation, namely $s_{1,2} = \frac{1\pm\sqrt{5}}{2}$, but it must be that S > 0, so only the biggest root satisfies this condition, and therefore $S = \frac{1+\sqrt{5}}{2}$ is by the way the biggest part of the "golden section".

The first person who combined mathematics and the geometrical aspects of a fractal was the Swedish mathematician Helge von Koch (1870-1924). He constructed the geometrical figure called the "Koch snowflake" by iterations. At the first step there is just an equilateral triangle.

To go to the next step, one has to split each side of the perimeter into three equal parts, to build on the equilateral triangle over the middle segment, as shown in the picture. So the figure at the second step is exactly the Star of David. From the third step onwards, the figure looks like a snowflake.

It turns out that, as the number of steps tends to infinity, the perimeter of a figure also tends to infinity but the area tends to o finite value, namely $\frac{8}{5}S_1$, where S_1 is the area of the original triangle. This fact was so unusual, that the Koch snowflake was nicknamed “The math freak”.



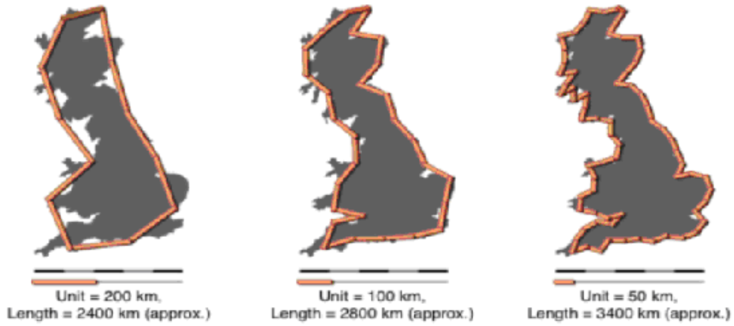
3. Benoit Mandelbrot and fractal theory

Benoit Mandelbrot (1924-2010) was a prominent mathematician who created and developed the mathematical theory of fractals. He had a phenomenal ability to make a clear geometrical interpretation of all the mathematical problems that he faced.

Benoit Mandelbrot

His first breakthrough in fractal theory was the solution of so-called coastline problem. This problem was described by Richardson (1960), who noted that there was a discrepancy between the coastline lengths reported by Portuguese and Spanish sources. Portugal stated that its border with Spain equals 987 km, while Spain defined it as equal to 1214 km. This difference appeared as due to the choice of “rulers” of different size to estimate the border’s length.

A similar problem appeared as geographers tried to measure Great Britain’s coastline. It turned out that, if the coastline is measured by 200 km segments, then the length equals 2300 km., if it is measured by 100 km. the result is 2800 km., if it is measured by 50 km. we obtain 3500 km. Of course, the measurement results differ significantly. What to do with that discrepancy and which result is true?



It was shown that none of measurement methods is true, and the coastline has no length at all in common meaning, just like the Koch snowflake (Mandelbrot, 1967). Moreover, Mandelbrot showed that the dimension of such lines is fractional: some number between 1 and 2. It turns out that the dimension of Great Britain's coastline equals D = 1.25, and for the frontier between Spain and Portugal it equals D = 1.14, and D = 1.13 for the Australian coastline. South Africa's coast, which appeared the smoothest in the atlas, has D = 1.02.

Let's have a look at Mandelbrot personality via some of his statements.

"The basic idea is that when you bring the fractal object closer, it continues to look the same".

"Clouds are not spheres, mountains are not cones, coastlines are not circles, and bark is not smooth, nor does lightning travel in a straight line."

"Mathematics describes a smooth world created by people. And the rough world created by nature turned out to be beyond the bounds of our mathematics."

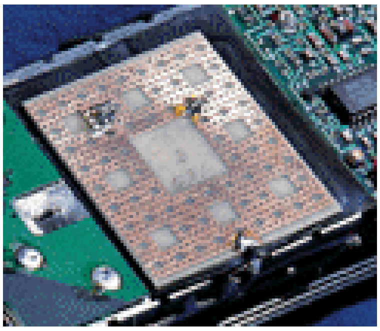
"For many years I had been hearing the comment that fractals make beautiful pictures, but are pretty useless. I was irritated because important applications always take some time to be revealed."

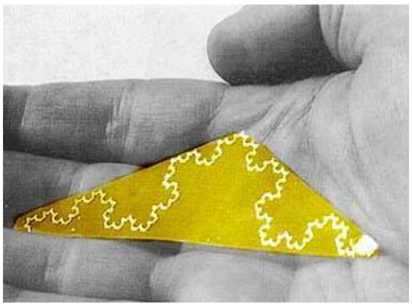
And applications came soon, one after another. In 1977 the first part of Star Wars, the American cult film series by George Lukas was released. Of course, the computing power at that time was much weaker than now, and so it was impossible to draw using computer graphic tools objects such as clouds and mountains so that they look believable. But the way out was found: the images of these objects were drawn as fractals.

The story of the invention of the fractal antenna sounds funny. Nathan Cohen, professor of Boston university in the field of physics and astronomy was a radio fan, and at the same time he had a big expensive antenna on the roof of his house in downtown Boston. One day he was forced to remove an antenna by city authorities because it spoiled the surrounding view. He was very disappointed and being desperate he made an antenna from aluminum foil and folded it in the shape of a Koch snowflake. Unexpectedly it turns out, this primitive tool worked better than the demounted brand antenna. He began to learn from this phenomenon and then changed the direction of his research. He constructed his first fractal antenna in 1988, and after the successful research effort he became a co-founder of the company Fractal Antenna Systems in 1995. This company produces the fractal antennas not only for TV, but also for mobile phones. Do you remember the first mobile phones with on antenna sticking out? Nathan Cohen's idea allowed the replacement of the sticking-out antenna with the fractal one, which may be hidden inside and this changed the design of mobile phones radically.

The Fractal Antenna Systems company is working successfully, developing modern technologies such as 5G ant IoT (internet of things).

Nathan Cohen





Retro mobile phone

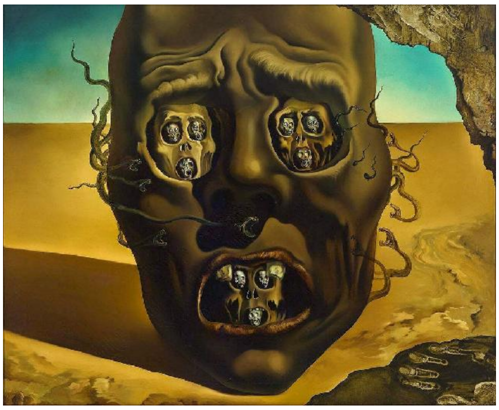
Fractal antennas for mobile phones

One of the obituaries about Mandelbrot contained the following phrase: “He was surrounded by fractals all of his life and even a cancer tumor, that killed him, had a fractal structure”. Sounds cynical? Not exactly. Actually, one of the rapidly-developing directions of modern oncology is based on fractal theory. It turns out that cancer tumors have a fractal structure that differs from the structure of both healthy tissue and benign tumors. Besides, the growth dynamics of benign and malignant tumors differ. These facts gives a background for principal new diagnostic tool creation that, we hope, will save many human lives.

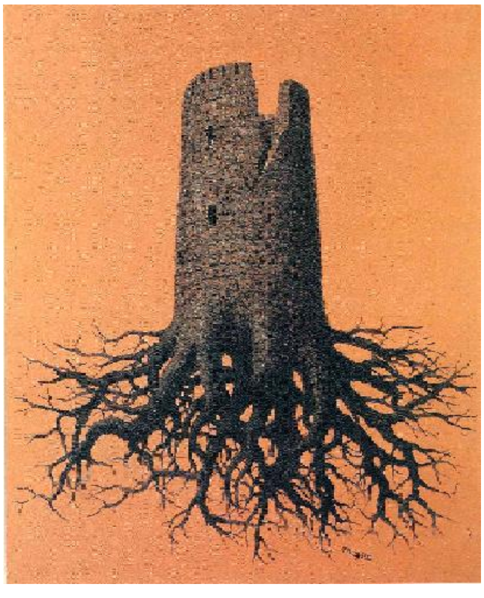
Fractals in painting

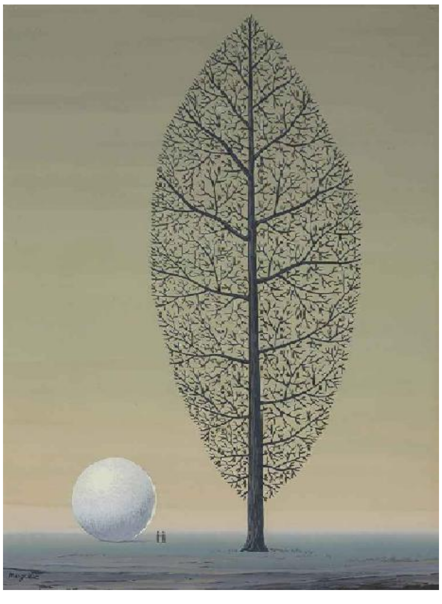
Without doubt, anyone can find a lot of fractal pictures on the internet. But in the author’s opinion, paintings that contains both fractal elements and some plot are the points of much more interest. Here are some examples.

Salvador Dali, “Face of war”, 1940. This picture was painted in 1940, at the beginning of the second world war. The war wasn’t in full swing yet, but its terrible essence was already revealed.



Two paintings by Rene Magritte, a Belgian surrealist (1898-1967)

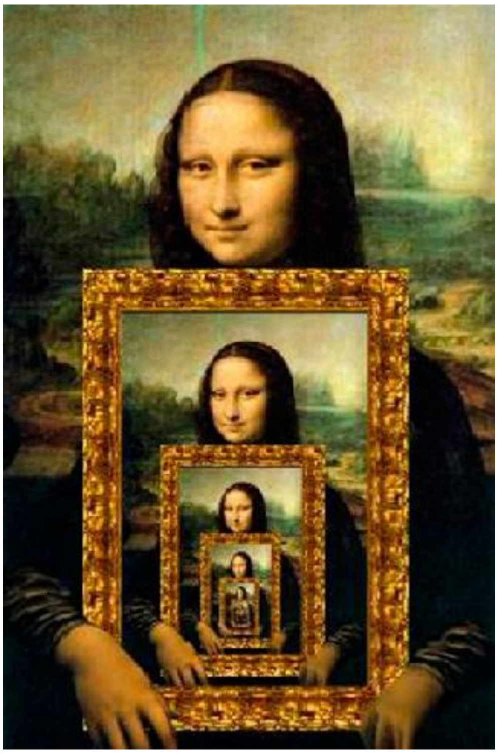




M. C. Escher, "Predestinations" (1951). M. C. Escher (1898-1972) was a Dutch graphic artist and master of optical illusions. The fishes and the ducks are depicted as complementary entities, predators, and preys. The future is pre-defined, as each duck is doomed to be eaten by some fish.

Vasya Lozhkin, born. 1976 (real name Aleksei Kudelin) is a famous modern Russian artist. The style of his paintings is primitivism, the plots are pranks using cats and ordinary people.

Two fractal pictures with no name.



Fractals in architecture

Fractals are represented in all forms of art and architecture is no exception. A natural question arises: what is the reason for constructing fractal structures? It is clear that all stages of fractal building construction require more effort and cost compared to the construction of ordinary (non-fractal) ones. But what do we get in return? For sure, the fractal structure becomes a landmark and an object of increased attention. In big cities it is one more sight that attracts tourists. For a small town, this is a great opportunity to stand out from a number of similar towns. For locals of big cities or small towns looking at such construction is the opportunity to break out of everyday life. Such buildings can be a good guide among monotonous buildings. For example, "I live three blocks left to fractal building" may be a good informal address. If a fractal construction is built as an office center, then some tenants are ready to overpay (even a few times more than the usual rent) for the sake of prestige and recognition of the office. In the case of a museum or theater, a larger number of visitors is expected.

It may be that some fractal construction, which is perceived now as strange, will receive universal approval later. There is a good example: namely the Eiffel tower. It was erected between 1887 and 1889 as an entry gate to the World Exhibition in Paris. However, the creative intelligentsia of Paris and France, led by the prominent architect Charles Garnier and including some of the most important figures in the arts, such as William-Adolphe Bouguereau and Guy de Maupassant, were outraged by Eiffel's daring project, and, starting from the very beginning, sent indignant demands to Paris City Hall to stop the construction of the tower. Writers and artists feared that the metal structure would overwhelm the architecture of the city, and violate the unique style of the capital that has developed over the centuries. Currently, the Eiffel Tower is undoubtedly the main tourist attraction and the main landmark of Paris.

It is noted that fractal architecture offers a new way to copy natural objects and induce natural forms into design (Salingaros, 2012); Taylor, 2006; Joye, 2007). So, an observer’s physiology may be influenced by contemplating fractal constructions. Anyway, fractal architecture may reduce the stress effect. Fractal constructions may use the stressful mood of excitement in the case of office centers, museums, theaters, casinos, etc. or to comfort in the case of churches, chapels, funeral houses, cemetery constructions, etc.

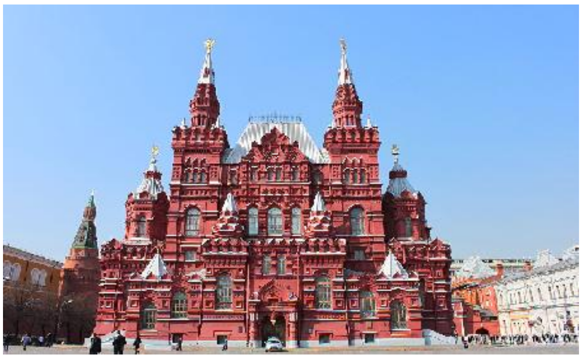
And one more principal question is: what kind of fractal structures seem to be more attractive than another and why?

According to Borda-de-Água (2019), the answer to the important question of why we should strive for fractal structures in architecture can probably be found in some intriguing studies on how human reacts to fractals. These studies were conducted by measuring physiological parameters and not by assessing verbal or written preferences, therefore concentrating on uncontrolled responses of the participants. Specifically, researchers used measures of skin conductance, which is known to be positively related with increased bodily stress. The participants were asked to performed several tasks that induce physiological stress, such as arithmetic calculations, while observing different images. The important result is that compared to a control non-fractal image, fractal images led to a reduction of the skin-conductance. Moreover, researchers found that the reduction was higher for middle range fractals, those that have a fractal dimension close to 1.5, that is, half-way between a smooth line (D=1) and a surface (D=2). As a justification to these results, Taylor (2006) points out that middle-range fractals are common in nature, such as in the contour of clouds or coastlines, indicating that this type of fractals are “a central feature of our daily visual experience”.

Some examples of fractal architecture.

Milano Cathedral (Duomo) and the Moscow Historical Museum are built in absolutely different architectural styles. But these buildings have something in common, namely they both show fractal structure details at all scales.

Milano Cathedral (Duomo)



Moscow Historical Museum

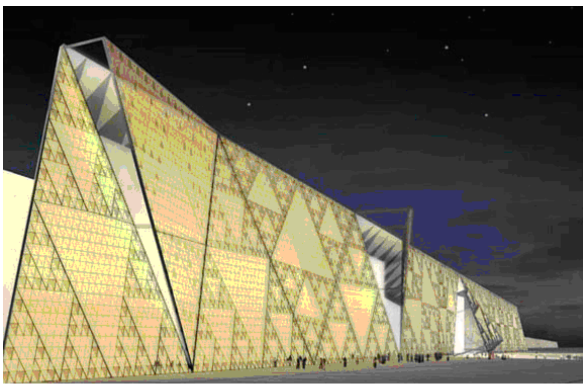
The temples of northern and southern India differ in style but both have fractal structures. The northern Indian temples are towers, surrounded by smaller towers, while the southern Indian temples are pyramids that consist of smaller pyramidal elements.

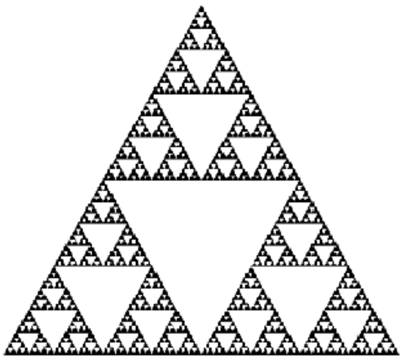
Northern Indian temple   Southern Indian temple

Here are some examples of modern fractal architecture.

“Storey Hall”, Melbourne, Australia.

The polygon tiles cover the facade and interior of Storey Hall. Such covering demonstrates 3-D analog of the so called Penrose tiling, which in 2-dimentional case consists of a set of non-overlapping polygons that cover the plane in an aperiodic way. That means that shifting any tiling by any distance, without rotation, cannot produce the same tiling.



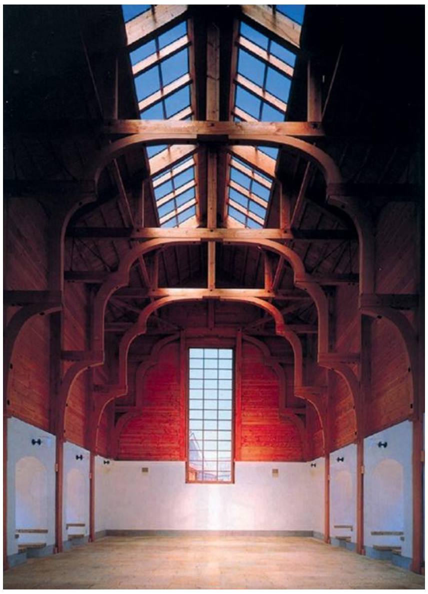


The Great Egyptian museum (or Giza museum), Giza, Egypt.

The facade is covered by large tiles in the shape of a fractal object, such as a Sierpinski triangle.

Stunning Japanese chapel, Nagasaki, Japan.

Fractal elements inside the chapel repeat the fractal structure of trees and so demonstrate the great design of the Creator.



Interior of the great hall, Eishin Campus, Tokyo.

The fractal design of the horseshoe-shaped roof gives an atmosphere of harmony and appeasement to visitors. The architect and design theorist Christopher Alexander, the author of “The Nature of Order” (Jiang, 2019) has a credo to design harmonious buildings in today's world.

City crematorium, Kyiv, Ukraine.

Fractal streamlined forms of the building give comfort and pacification to the relatives of the deceased.

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